

Super-Acceleration from Massless, Minimally Coupled ϕ^4 V. K. Onemli[†]
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ABSTRACT

We derive a simple form for the propagator of a massless, minimally coupled scalar in a locally de Sitter geometry of arbitrary spacetime dimension. We then employ it to compute the fully renormalized stress tensor at one and two loop orders for a massless, minimally coupled ϕ^4 theory which is released in Bunch-Davies vacuum at $t = 0$ in co-moving coordinates. In this system the uncertainty principle elevates the scalar above the minimum of its potential, resulting in a phase of super-acceleration. With the non-derivative self-interaction the scalar's breaking of de Sitter invariance becomes observable. It is also worth noting that the weak energy condition is violated on cosmological scales. An interesting subsidiary result is that canceling overlapping divergences in the stress tensor requires a conformal counterterm which has no effect on purely scalar diagrams.

PACS numbers: 04.62.+v, 98.80.Cq, 98.80.Hw

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1 Introduction

One of the most intriguing and potentially important results of observational cosmology is the inference, from Type Ia supernovae at high redshift [1, 2], that the universe is entering a phase of “accelerated expansion”. This phrase sometimes gives rise to a misconception that is best explained in the context of a homogeneous, isotropic and spatially flat geometry,

$$ds^2 = -dt^2 + a^2(t)d\vec{x} \cdot d\vec{x} . \quad (1)$$

The rate of cosmological expansion is the Hubble constant,

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)} , \quad (2)$$

whereas the deceleration parameter is,

$$q(t) \equiv -1 - \frac{\dot{H}(t)}{H^2(t)} . \quad (3)$$

Although the general public sometimes takes “accelerated expansion” to mean that $\dot{H} > 0$, the actual meaning is rather that $q < 0$.

In fact no stable theory can exhibit $\dot{H} > 0$ on the classical level. This is a simple consequence of the two nontrivial Einstein equations in this geometry,

$$3H^2 = 8\pi G\rho , \quad (4)$$

$$-2\dot{H} - 3H^2 = 8\pi Gp . \quad (5)$$

Here $\rho(t)$ and $p(t)$ are the energy density and pressure, respectively. Adding the two equations gives,

$$-2\dot{H} = 8\pi G(\rho + p) . \quad (6)$$

The quantity on the right-hand side is non-negative as a consequence of the Weak Energy Condition [3] which asserts that no physical observer can see a negative energy density. Its mathematical transcription is $T_{\mu\nu}W^\mu W^\nu \geq 0$ for all timelike vectors W^μ . To derive $\rho + p \geq 0$ for a homogeneous and isotropic cosmology, note first that the nonzero elements of the stress-energy tensor are $T_{00} = -\rho g_{00}$ and $T_{ij} = pg_{ij}$. Now apply the condition in the limit that W^μ becomes null.

The Weak Energy Condition not only implies $\dot{H} \leq 0$, it also tells us that $q \geq -1$. Although stable theories obey the Weak Energy Condition on the classical level it has long been known that quantum effects can give rise to violations [4], although those studied so far have been on microscopic scales. This is of great interest because the present data is actually consistent with $q < -1$ [5]. It is only theoretical prejudice in favor of stability and against quantum effects on cosmological scales that results in the usual likelihood contours being cut off at $q = -1$. If the proposed Supernova Pencil Beam Survey [6] or SNAP experiments [7] were to give a definitive determination of $q < -1$ we should be forced to abandon this prejudice. The purpose of this paper is to give a precise formulation of a field theory in which quantum cosmological effects do in fact result in $\rho + p < 0$ for an arbitrarily long period of time.

The model has been studied before, as has its potential for violating the Weak Energy Condition on cosmological scales [8]. It consists of a massless, minimally coupled scalar with a quartic self-interaction. Although gravity is not dynamical, neither is the background flat. The scalar is a spectator to Λ -driven inflation. The new feature, which is the subject of this paper, is that we can now regulate the model in a simple way that preserves general coordinate invariance. The acid test of simplicity is that one can go beyond coincident propagators to explicitly evaluate higher loop graphs which involve integrations. We shall demonstrate this by computing all one and two loop contributions to the expectation value of the stress-energy tensor in the presence of a state which is Bunch-Davies vacuum at $t = 0$. Our result confirms the conjecture of ref. [8] that this model shows super-acceleration, i.e., $\rho + p < 0$. The model should also be of interest as an exercise in quantum field theory on curved spacetime in which the computations are pushed beyond the one-loop approximation that has been so thoroughly studied [9].

This Introduction is the first of eight sections. In Section 2 we present the Lagrangian and describe the various diagrams which comprise the expectation value of the stress-energy tensor at one and two loop order. Section 3 gives an explicit and relatively simple expression for the D -dimensional propagator, the possession of which is what allows us to apply dimensional regularization at arbitrary order. In Section 4 we compute the one loop mass counterterm and the one loop expectation value of the stress-energy tensor. Section 5 evaluates the leading order, local diagrams that were already obtained in ref. [8]. The power of our dimensional regularization technique is

displayed in Section 6 by computing the nonlocal diagrams which could not be done previously. Although these terms do not contribute at leading order (as conjectured previously [8]) they are necessary to make the stress-energy tensor conserved and they do probe the nonlocal, ultraviolet finite sector of the theory. They also contribute some nonlocal ultraviolet divergences which, if uncompensated, would compromise the model's renormalizability. Section 7 evaluates the expectation value of the conformal counterterm which absorbs them. (The need for this was also realized before [8].) The fully renormalized result is presented in Section 8.

2 The model and its stress-energy tensor

Our model has the following Lagrangian,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu}\sqrt{-g} - \frac{1}{4!}\lambda\phi^4\sqrt{-g} + \Delta\mathcal{L} . \quad (7)$$

The counterterms reside in $\Delta\mathcal{L}$,

$$\begin{aligned} \Delta\mathcal{L} = & -\frac{1}{2}\delta m^2\phi^2\sqrt{-g} + \delta\xi\left(R - D(D-1)H^2\right)\phi^2\sqrt{-g} - \frac{\delta\Lambda}{8\pi G}\sqrt{-g} , \\ & -\frac{1}{2}\delta Z\partial_\mu\phi\partial_\nu\phi g^{\mu\nu}\sqrt{-g} - \frac{1}{4!}\delta\lambda\phi^4\sqrt{-g} . \end{aligned} \quad (8)$$

Those on the second line are of order λ^2 , just as in flat space [10], and are irrelevant for the purposes of this paper. However, the counterterms on the top line are needed to remove divergences at one and two loop order in the stress-energy tensor. It turns out that δm^2 is of order λ while $\delta\Lambda$ is actually of order λ^0 . The conformal counterterm, $\delta\xi$ is needed to remove overlapping divergences in the expectation value of $T_{\mu\nu}$ at order λ^1 . Note that this term makes no contribution at all to pure scalar interactions on account of the fact that $R = D(D-1)H^2$ in D -dimensional, locally de Sitter background.

Only the scalar is quantized. The metric is a non-dynamical background which we take to be locally de Sitter geometry with cosmological constant $\Lambda = (D-1)H^2$. It is most convenient to work in conformal coordinates for which the metric takes the form,

$$g_{\mu\nu}(\eta, \vec{x}) = \Omega^2(\eta)\eta_{\mu\nu} \quad \text{where} \quad \Omega(\eta) = -\frac{1}{H\eta} . \quad (9)$$

It is useful to keep in mind the relation to co-moving coordinates,

$$\eta = -H^{-1}e^{-Ht} \quad \Leftrightarrow \quad t = H^{-1} \ln(\Omega(\eta)) . \quad (10)$$

We release the state in Bunch-Davies vacuum at $t = 0$, corresponding to conformal time $\eta = -H^{-1}$. Note that the infinite future corresponds to $\eta \rightarrow 0^-$, so the possible variation of causally related conformal coordinates in either space or time is at most $\Delta x = \Delta \eta = H^{-1}$.

The stress-energy tensor is,

$$T_{\mu\nu}(x) \equiv \frac{-2}{\sqrt{-g(x)}} \frac{\delta S_{matter}}{\delta g^{\mu\nu}(x)} , \quad (11)$$

$$\begin{aligned} &= (1 + \delta Z) \left[\delta^\rho_\mu \delta^\sigma_\nu - \frac{1}{2} g_{\mu\nu}(x) g^{\rho\sigma}(x) \right] \partial_\rho \phi(x) \partial_\sigma \phi(x) \\ &\quad - g_{\mu\nu}(x) \frac{\lambda + \delta\lambda}{4!} \phi^4(x) - \frac{1}{2} \delta m^2 g_{\mu\nu}(x) \phi^2(x) - \frac{\delta\Lambda}{8\pi G} g_{\mu\nu} \\ &\quad - 2\delta\xi \left[(D-1)H^2 \phi^2 g_{\mu\nu} + g_{\mu\nu} (\phi^2)^{;\rho}{}_\rho - (\phi^2)_{;\mu\nu} \right] . \end{aligned} \quad (12)$$

We want the one and two loop contributions to the expectation value of this operator in the state which is Bunch-Davies vacuum at $t = 0$. If $i\Delta(x; x')$ stands for the bare propagator in this state, the lowest order kinetic energy contributions are,

$$\begin{aligned} &\left[\delta^\rho_\mu \delta^\sigma_\nu - \frac{1}{2} g_{\mu\nu}(x) g^{\rho\sigma}(x) \right] \langle \Omega | \partial_\rho \phi(x) \partial_\sigma \phi(x) | \Omega \rangle \\ &= \left[\delta^\rho_\mu \delta^\sigma_\nu - \frac{1}{2} g_{\mu\nu}(x) g^{\rho\sigma}(x) \right] \left\{ \partial_\rho \partial'_\sigma i\Delta(x; x') \Big|_{x' \rightarrow x} + O(\lambda) \right\} . \end{aligned} \quad (13)$$

The first term is evaluated in Section 4 and the order λ correction is computed in Section 6. The lowest order contributions from the potential energy are,

$$\begin{aligned} &-g_{\mu\nu}(x) \langle \Omega | \frac{\lambda}{4!} \phi^4(x) + \frac{\delta m^2}{2} \phi^2(x) | \Omega \rangle \\ &= -g_{\mu\nu}(x) \left\{ \frac{\lambda}{8} [i\Delta(x; x)]^2 + \frac{\delta m^2}{2} i\Delta(x; x) + O(\lambda^2) \right\} . \end{aligned} \quad (14)$$

This is evaluated in Section 5. The contribution from the conformal counterterm is,

$$-2\delta\xi \left\{ (D-1)H^2 g_{\mu\nu}(x) \langle \Omega | \phi^2(x) | \Omega \rangle \right.$$

$$\begin{aligned}
& + \left[g_{\mu\nu}(x)g^{\rho\sigma}(x) - \delta^\rho_\mu \delta^\sigma_\nu \right] \left(\partial_\rho \partial_\sigma - \Gamma^\alpha_{\rho\sigma}(x) \partial_\alpha \right) \langle \Omega | \phi^2(x) | \Omega \rangle \} \\
& = -2\delta\xi \left\{ (D-1)H^2 g_{\mu\nu}(x) i\Delta(x; x) \right. \\
& \quad \left. + \left[g_{\mu\nu}(x)g^{\rho\sigma}(x) - \delta^\rho_\mu \delta^\sigma_\nu \right] \left(\partial_\rho \partial_\sigma - \Gamma^\alpha_{\rho\sigma}(x) \partial_\alpha \right) i\Delta(x; x) + O(\lambda^2) \right\}. \quad (15)
\end{aligned}$$

This term is evaluated in Section 7.

Obtaining a true expectation value, rather than an in-out matrix element, requires use of the Schwinger-Keldysh formalism [11, 12]. However, this formalism reduces to the usual Feynman diagrams for the lowest order graphs. We shall not actually need the full Schwinger-Keldysh formalism until discussing order λ kinetic energy corrections in Section 5.

Because the state is homogeneous and isotropic, the nonzero tensor components are $\langle \Omega | T_{00} | \Omega \rangle = -\rho g_{00}$ and $\langle \Omega | T_{ij} | \Omega \rangle = p g_{ij}$. This is true as well for each of the three contributions just described, so we shall finally report an energy density and pressure for each. Although neither the kinetic energy nor the potential energy contributions is separately conserved, their sum is. The conformal contribution is conserved by itself, as is the cosmological counterterm. However, because these quantities harbor divergences, “conservation” must be understood with the ultraviolet regulator still on. That is, we must keep the spacetime dimension D arbitrary, which makes conservation read, $\dot{\rho} = -(D-1)H(\rho + p)$. This is an important check on accuracy. Another important check is the cancellation of overlapping divergences that occurs when the various contributions are combined.

3 The D -dimensional propagator

The behavior of free massless minimally coupled scalar field has been investigated extensively [13, 14, 15, 16, 17, 18]. Among the curious properties of these particles are the absence of normalizable, de Sitter invariant states [13] and the appearance of acausal infrared singularities when the Bunch-Davies vacuum is used with infinite spatial surfaces [14, 15]. To regulate this infrared problem we work on the manifold $T^{D-1} \times R$, with the spatial coordinates in the finite range, $-H^{-1}/2 < x^i \leq H^{-1}/2$. Although the actual propagator is a mode sum on this manifold, the small possible variation in conformal coordinates renders the first term of the Euler-Maclaurin formula — just the integral — an excellent approximation, and the finite spatial range serves

merely to cut off what would have been a logarithmic infrared divergence on infinite space. In $D = 3 + 1$ spacetime dimensions the result is [19],

$$i\Delta(x; x')|_{D=4} = \left(\frac{H}{2\pi}\right)^2 \left\{ \frac{1}{y(x; x')} - \frac{1}{2} \ln(y(x; x')) + \frac{1}{2} \ln(\Omega(\eta)\Omega(\eta')) \right\} , \quad (16)$$

where the modified de Sitter length function is,¹

$$y(x; x') \equiv \Omega(\eta)\Omega(\eta')H^2 \left[\|\vec{x} - \vec{x}'\|^2 - (|\eta - \eta'| - ie)^2 \right] . \quad (17)$$

Neglecting the higher order Euler-Maclaurin terms does not prevent (16) from solving the correct differential equation. The higher terms also drop out of quite complicated, nonlinear relations such as the Ward Identity for the one loop graviton self-energy [20]. We shall therefore regard the technique as valid and confine ourselves to finding the appropriate generalization of (16) to D spacetime dimensions.

We seek a function of $y(x; x')$ and the two conformal factors which obeys the equation,

$$\eta^{\mu\nu} \frac{\partial}{\partial x^\mu} \Omega^{D-2}(\eta) \frac{\partial}{\partial x^\nu} i\Delta(x; x') = i\delta^D(x - x') . \quad (18)$$

One consequence of de Sitter invariance is that the kinetic operator takes a particularly simple form when acting on a function of just $y(x; x')$. To reach this form note first that derivatives of $y(x; x')$ give,

$$\frac{\partial y}{\partial x^\mu} = H\Omega(\eta) \left[y\delta_\mu^0 + 2\Omega(\eta')H\Delta x_\mu + 2ieH\Omega(\eta')\text{sgn}(\eta - \eta')\delta_\mu^0 \right] , \quad (19)$$

$$\begin{aligned} \frac{\partial^2 y}{\partial x^\mu \partial x^\nu} = & H^2\Omega^2(\eta) \left[2y\delta_\mu^0\delta_\nu^0 + 2\Omega(\eta')H\Delta x_\mu\delta_\nu^0 + 2\Omega(\eta')\delta_\mu^0H\Delta x_\nu + 2\frac{\Omega(\eta')}{\Omega(\eta)}\eta_{\mu\nu} \right. \\ & \left. + 4ieH\Omega(\eta')\text{sgn}(\eta - \eta')\delta_\mu^0\delta_\nu^0 + 4ie\delta(\eta - \eta')\delta_\mu^0\delta_\nu^0 \right] . \end{aligned} \quad (20)$$

¹What is termed “the de Sitter length function” in the literature is,

$$z(x; x') = 1 - y(x; x') .$$

The geodesic length from x^μ to x'^μ , $\ell(x; x')$, is related to $y(x; x')$ as follows,

$$y(x; x') = \sin^2 \left(\frac{1}{2} H\ell(x; x') \right) .$$

Now use $\Omega(\eta')H\Delta x^0 = 1 - \Omega(\eta')/\Omega(\eta)$ to conclude,

$$\eta^{\mu\nu}\delta^0_\mu\frac{\partial y}{\partial x^\nu} = H\Omega(\eta)\left[-y + 2 - 2\frac{\Omega(\eta')}{\Omega(\eta)} - 2ieH\Omega(\eta')\text{sgn}(\eta - \eta')\right], \quad (21)$$

$$\eta^{\mu\nu}\frac{\partial y}{\partial x^\mu}\frac{\partial y}{\partial x^\nu} = H^2\Omega^2(\eta)\left[-y^2 + 4y - 4ieH\Omega(\eta')y\text{sgn}(\eta - \eta')\right], \quad (22)$$

$$\begin{aligned} \eta^{\mu\nu}\frac{\partial^2 y}{\partial x^\mu\partial x^\nu} &= H^2\Omega^2(\eta)\left[-2y + 4 + 2(D-2)\frac{\Omega(\eta')}{\Omega(\eta)} \right. \\ &\quad \left. - 4ieH\Omega(\eta')\text{sgn}(\eta - \eta') - 4ie\delta(\eta - \eta')\right]. \end{aligned} \quad (23)$$

Finally, use the chain rule and substitute (21-23),

$$\begin{aligned} \eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\Omega^{D-2}(\eta)\frac{\partial}{\partial x^\nu}f(y(x; x')) \\ = \Omega^{D-2}(\eta)\eta^{\mu\nu}\left\{\frac{\partial y}{\partial x^\mu}\frac{\partial y}{\partial x^\nu}f''(y) + \frac{\partial^2 y}{\partial x^\mu\partial x^\nu}f'(y) \right. \\ \left. + (D-2)H\Omega(\eta)\delta^0_\mu\frac{\partial y}{\partial x^\nu}f'(y)\right\}, \end{aligned} \quad (24)$$

$$\begin{aligned} = H^2\Omega^D(\eta)\left\{(4y - y^2)f''(y) + D(2 - y)f'(y) - 4ie\delta(\eta - \eta')f'(y) \right. \\ \left. - 2ieH\Omega(\eta')\text{sgn}(\eta - \eta')[2yf''(y) + Df'(y)]\right\}. \end{aligned} \quad (25)$$

Before considering possible dependence upon the scale factor let us note that the delta function on the right hand side of (18) descends from a factor of $y^{1-\frac{D}{2}}$ in the limit that $e \rightarrow 0$. To see this, suppose $f(y)$ has the form,

$$f(y) = \frac{k_1}{y^{\frac{D}{2}-1}} + O\left(y^{-\frac{D}{2}}\right), \quad (26)$$

where k_1 is a constant. Acting the kinetic operator gives,

$$\partial^\mu\Omega^{D-2}\partial_\mu f(y) = H^2\Omega^D\left\{-4ie\delta(\eta - \eta')\frac{-k_1(\frac{D}{2}-1)}{y^{\frac{D}{2}}} + O\left(y^{1-\frac{D}{2}}\right)\right\}. \quad (27)$$

Now multiply this by a test function (of x^μ), integrate $\int d^D x$, and take $e \rightarrow 0$. For $e = 0$, terms of order $y^{1-\frac{D}{2}}$ diverge at $x^\mu = x'^\mu$, but the singularity is

integrable. The only delta function comes from the term proportional to,

$$e\delta(\eta - \eta') \frac{1}{y^{\frac{D}{2}}(x; x')} = \frac{\delta(\eta - \eta')}{H^D \Omega^D} \frac{e}{(\|\vec{x} - \vec{x}'\|^2 + e^2)^{\frac{D}{2}}} \longrightarrow \frac{\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} \frac{\delta^D(x - x')}{H^D \Omega^D}. \quad (28)$$

Comparison with (18) fixes k_1 ,

$$k_1 = \left(\frac{H}{2\pi}\right)^2 \left(\frac{\pi}{H^2}\right)^{2-\frac{D}{2}} \Gamma\left(\frac{D}{2} - 1\right). \quad (29)$$

Now consider adding to $f(y)$ a symmetric function of $\Omega(\eta)$ and $\Omega(\eta')$. The only function which can give the same prefactor of $\Omega^D(\eta)$ is a constant times the same one that appears in (16). The D -dimensional propagator must therefore take the form,

$$i\Delta(x; x') = f(y(x; x')) + k_2 \ln(\Omega(\eta)\Omega(\eta')), \quad (30)$$

In the $e \rightarrow 0$ limit the function $f(y)$ must obey,

$$H^2 \Omega^D(\eta) \left\{ (4y - y^2) f''(y) + D(2 - y) f'(y) - 4ie\delta(\eta - \eta') f'(y) - k_2(D - 1) \right\} = i\delta^D(x - x'). \quad (31)$$

We have seen that getting the correct delta function requires a term of the form $y^{1-\frac{D}{2}}$ plus less singular powers. Series solution generates the higher powers. Defining $D \equiv 4 - \epsilon$ gives,

$$\frac{H^{2-\epsilon}}{4\pi^{2-\frac{\epsilon}{2}}} \Gamma\left(1 - \frac{\epsilon}{2}\right) \left\{ \frac{1}{y^{1-\frac{\epsilon}{2}}} - \left(1 - \frac{\epsilon}{2}\right) \sum_{n=0}^{\infty} \frac{1}{n + \frac{\epsilon}{2}} \frac{\Gamma(3 + n - \frac{\epsilon}{2})}{(n + 1)! \Gamma(2 - \frac{\epsilon}{2})} \frac{y^{n+\frac{\epsilon}{2}}}{4^{n+1}} \right\}. \quad (32)$$

This series solves (31), for $k_2 = 0$, but it does not reduce to (16) for $\epsilon = 0$. The $n = 0$ term is not even finite in this limit! The resolution to both problems is a series of strictly nonnegative integer powers of y , which cancels the divergence and the unwanted terms. This series obeys the homogeneous equation up to a y^0 term which is canceled by the $k_2(D - 1)$ term. The final result is,

$$i\Delta(x; x') = \left(\frac{H}{2\pi}\right)^2 \left(\frac{H}{\sqrt{\pi}}\right)^{-\epsilon} \Gamma\left(1 - \frac{\epsilon}{2}\right) \left\{ \frac{1}{y^{1-\frac{\epsilon}{2}}} + \left(1 - \frac{\epsilon}{2}\right) \left(1 - \frac{\epsilon}{4}\right) \left(\frac{1 - y^{\frac{\epsilon}{2}}}{\epsilon}\right) \right\}$$

$$+ \left(1 - \frac{\epsilon}{2}\right) \sum_{n=1}^{\infty} \left[\frac{1}{n} \frac{\Gamma(3+n-\epsilon)}{\Gamma(2+n-\frac{\epsilon}{2})} - \frac{1}{n+\frac{\epsilon}{2}} \frac{\Gamma(3+n-\frac{\epsilon}{2})}{(n+1)!\Gamma(2-\frac{\epsilon}{2})} y^{\frac{\epsilon}{2}} \right] \frac{y^n}{4^{n+1}} + \frac{1}{4} \frac{\Gamma(3-\epsilon)}{\Gamma(1-\frac{\epsilon}{2})} \ln(\Omega(\eta)\Omega(\eta')) \Big\} . \quad (33)$$

The great advantage of this regularization is that it preserves general coordinate invariance (once ϵ is taken to zero). One might think that the propagator is unwieldy but this is not so in practice. The really cumbersome part is the infinite sum on the second line. But these terms all vanish at coincidence ($y(x; x) = 0$) and they vanish for all $y(x; x')$ at $D = 4$. So one need only retain the higher terms when they multiply something else that diverges like $1/\epsilon$. Note also that one need never worry about large $y(x; x')$ on account of causality.

All valid regularizations must reproduce the result of Vilenkin and Ford that the coincidence limit of the propagator contains a finite term which grows like $\ln(\Omega) = Ht$ [16, 17, 18]. To check this note that $y(x; x) = 0$ at coincidence. When a variable vanishes like this in dimensional regularization one must always assume ϵ to be large enough that the variable is raised to only nonnegative powers. We therefore find,

$$\lim_{x' \rightarrow x} i\Delta(x; x') = \left(\frac{H}{2\pi}\right)^2 \left(\frac{H}{\sqrt{\pi}}\right)^{-\epsilon} \left\{ \frac{1}{2\epsilon} \Gamma\left(3 - \frac{\epsilon}{2}\right) + \frac{1}{2} \Gamma(3 - \epsilon) \ln(\Omega(\eta)) \right\} . \quad (34)$$

Note that (34) is exact for *arbitrary* ϵ .

For every sort of line in the standard Feynman rules the Schwinger-Keldysh formalism has a “+” line and a “−” line of the same sort [11, 12]. Although vertices involve either all + or all − lines, there are ++, +−, −+ and −− propagators. All of them are the same function (33) of the appropriate version of the modified de Sitter length function,

$$y_{++}(x; x') \equiv \Omega(\eta)\Omega(\eta')H^2 \left[\|\vec{x} - \vec{x}'\|^2 - (|\eta - \eta'| - i\epsilon)^2 \right] , \quad (35)$$

$$y_{+-}(x; x') \equiv \Omega(\eta)\Omega(\eta')H^2 \left[\|\vec{x} - \vec{x}'\|^2 - (\eta - \eta' + i\epsilon)^2 \right] , \quad (36)$$

$$y_{-+}(x; x') \equiv \Omega(\eta)\Omega(\eta')H^2 \left[\|\vec{x} - \vec{x}'\|^2 - (\eta - \eta' - i\epsilon)^2 \right] , \quad (37)$$

$$y_{--}(x; x') \equiv \Omega(\eta)\Omega(\eta')H^2 \left[\|\vec{x} - \vec{x}'\|^2 - (|\eta - \eta'| + i\epsilon)^2 \right] . \quad (38)$$

When the points are spacelike related $y(x; x')$ is positive; when they are timelike $y(x; x')$ is negative. So $i\Delta_{++}(x; x')$ and $i\Delta_{+-}(x; x')$ are equal (in the

limit that e vanishes) for all spacelike separated points and for $\eta' > \eta$. That is why the $++$ and $+-$ contributions cancel whenever $x^{\mu'}$ strays outside the past lightcone of x^μ . Note that inside the past lightcone the $++$ and $+-$ propagators are conjugate.

4 The one loop counterterms

The one loop (λ^0) kinetic energy contribution from (13) provides a fine illustration of how (33) is used to compute dimensionally regulated results. Since $y(x; x') \sim \Omega(\eta)\Omega(\eta')H^2(x-x')^2$ vanishes at coincidence, only the $n = 1$ term can contribute after the action of two derivatives,²

$$\partial_\rho \partial'_\sigma i\Delta(x; x') \Big|_{x' \rightarrow x} = -\frac{H^4}{2^5 \pi^2} \left(\frac{H}{\sqrt{\pi}} \right)^{-\epsilon} \frac{\Gamma(4-\epsilon)}{2-\frac{\epsilon}{2}} g_{\rho\sigma}(x) . \quad (39)$$

Contracting with the tensor prefactor in (13) gives the one loop kinetic energy contribution,

$$\frac{H^4}{2^5 \pi^2} \left(\frac{H}{\sqrt{\pi}} \right)^{-\epsilon} \frac{(1-\frac{\epsilon}{2})\Gamma(4-\epsilon)}{2-\frac{\epsilon}{2}} g_{\mu\nu}(x) . \quad (40)$$

Although ultraviolet finite, this can be nulled by the following cosmological counterterm,

$$\delta\Lambda = \frac{GH^4}{4\pi} \left(\frac{H}{\sqrt{\pi}} \right)^{-\epsilon} \frac{(1-\frac{\epsilon}{2})\Gamma(4-\epsilon)}{2-\frac{\epsilon}{2}} + O(\lambda) . \quad (41)$$

Fig. 1 depicts the one loop scalar self mass-squared:

$$-iM_{1\text{-loop}}^2(x; x') = -i \left[\frac{\lambda}{2} i\Delta(x; x) + \delta m^2 \right] \delta^D(x - x') . \quad (42)$$

It is calculated by using the coincidence limit of the propagator given in Eq. (34). Because of the finite, time-dependent term we cannot make the self

²Since the time-ordering commutator term should not really be present we avoid the delta function term by first evaluating for $x^{\mu'} \neq x^\mu$ and then taking coincidence. Another way of getting the same result would be to use the $-+$ propagator.

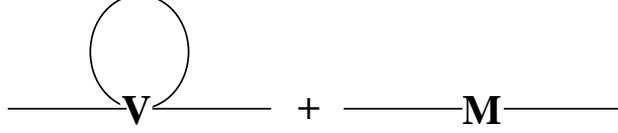


Figure 1: The scalar self mass-squared at order λ . V denotes the 4-point vertex and M stands for the mass counterterm vertex.

mass-squared vanish for all time. A reasonable renormalization condition is that it should be zero at $t = 0$. This can be enforced by,

$$\delta m^2 = -\frac{\lambda H^{2-\epsilon}}{2^4 \pi^{2-\epsilon/2}} \frac{1}{\epsilon} \Gamma\left(3 - \frac{\epsilon}{2}\right) + O(\lambda^2) . \quad (43)$$

The resulting renormalized self mass-squared is,

$$M_{1\text{-loop}}^2(x; x') = \frac{\lambda}{8\pi^2} H^2 H t \delta^D(x - x') . \quad (44)$$

The temporal growth of M^2 has a straightforward physical interpretation. It is only at $\phi = 0$ that the scalar potential has zero curvature; for a general field configuration the curvature goes like ϕ^2 . But we know from (34) that the expectation value of ϕ^2 grows with time as the scalar executes a drunkard's walk. Hence the mass-squared must exhibit the same time dependence.

5 The potential energy contributions

The two loop contribution from the potential energy is depicted in Fig. 2. A simple application of the Feynman rules reveals it to be,

$$T_{\mu\nu}^P(x) = -g_{\mu\nu}(x) \left\{ \frac{\lambda}{8} [i\Delta(x; x)]^2 + \frac{\delta m^2}{2} i\Delta(x; x) \right\} . \quad (45)$$

Substituting our results (34) for the coincidence limit and (43) for the mass counterterm gives,

$$T_{\mu\nu}^P(x) = g_{\mu\nu}(x) \frac{\lambda H^4}{2^6 \pi^4} \left\{ \left(\frac{\pi}{H^2} \right)^\epsilon \frac{\Gamma^2(3 - \frac{\epsilon}{2})}{8\epsilon^2} - \frac{1}{2} \ln^2(\Omega(\eta)) \right\} . \quad (46)$$

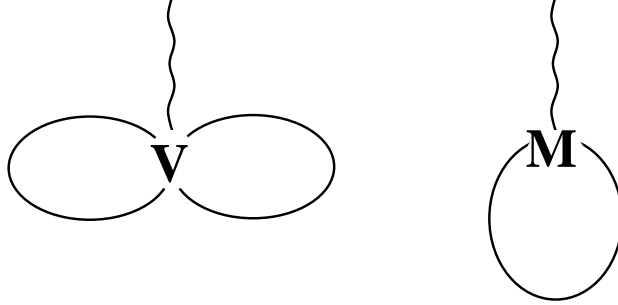


Figure 2: The potential energy contributions to the scalar stress-energy tensor at order λ . V denotes the 4-point vertex and M stands for the mass counterterm vertex.

Since $T_{00} \equiv -\rho g_{00}$, and $T_{ij} \equiv p g_{ij}$, the energy density ρ and pressure p are,

$$\rho_P = -p_P = \frac{\lambda H^4}{2^6 \pi^4} \left\{ - \left(\frac{\pi}{H^2} \right)^\epsilon \frac{\Gamma^2(3 - \frac{\epsilon}{2})}{8\epsilon^2} + \frac{1}{2} \ln^2(\Omega(\eta)) \right\}. \quad (47)$$

The temporal growth of these terms has the same physical interpretation as that of the mass-squared. Since the scalar changes slowly we expect that the kinetic energy contributions are subdominant. The next section will confirm this expectation.

6 Order λ kinetic contributions

Fig. 3 depicts the order λ contributions from the kinetic energy. It is only these graphs which require the full Schwinger-Keldysh formalism, very readable derivations of which exist in the literature [11, 12]. The rules themselves are simple. For every kind of vertex that might appear in a Feynman diagram the Schwinger-Keldysh formalism has two kinds of vertices: a $+$ vertex which is the same as that of Feynman, and a $-$ vertex which is the conjugate. Propagation can take place between any two kinds of vertices using the appropriate propagator. Each propagator is the same function of the four variables $y_{\pm\pm}(x; x')$ given at the end of Section 3.

Using the Schwinger-Keldysh formalism is straightforward. One simply draws the analogous Feynman diagram and then sums over \pm variations. If

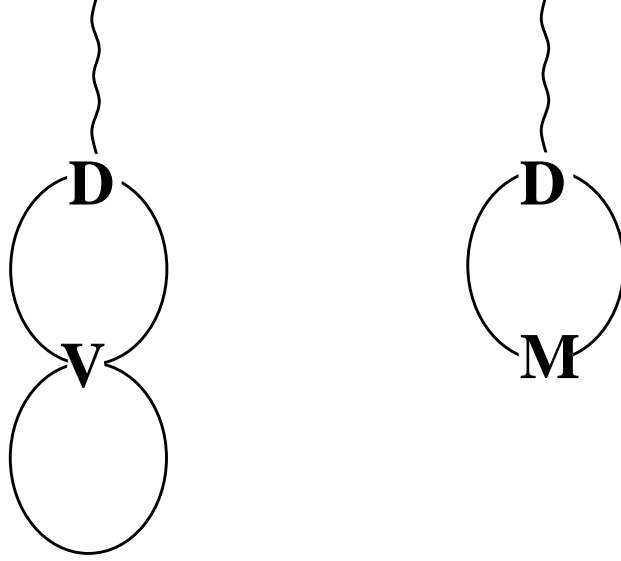


Figure 3: Order λ kinetic energy contributions to the scalar stress-energy tensor. The derivative vertex is D , V denotes the 4-point vertex and M stands for the mass counterterm vertex.

the operator under study is to be time-ordered (as in our case) then the external lines are $+$. For the diagrams of Fig. 3 this means that the vertex marked D is fixed to be $+$, however one must sum over both $+$ and $-$ contributions in the V and M vertices, of course using the appropriate propagators. The result is,

$$T_{\mu\nu}^K(x) = \left[\delta_\mu{}^\rho \delta_\nu{}^\sigma - \frac{1}{2} \eta_{\mu\nu} \eta^{\rho\sigma} \right] \int d^D x' \Omega^D(\eta') \left\{ \partial_\rho i\Delta_{++}(x; x') \partial_\sigma i\Delta_{++}(x; x') \right. \\ \left. - \partial_\rho i\Delta_{+-}(x; x') \partial_\sigma i\Delta_{+-}(x; x') \right\} \times \left\{ -\frac{i}{2} \lambda i\Delta(x'; x') - i\delta m^2 \right\}. \quad (48)$$

Since both diagrams of Fig. 3 have the same upper loop, they possess a common factor in the first curly bracket. The first term within the final curly bracket derives from the left hand diagram, while the second term comes from the right hand diagram.

It is convenient to subsume the complicated, ϵ -dependent constants which appear in the propagator (33),

$$i\Delta(x; x') \equiv \alpha \left\{ \gamma(y(x; x')) + \beta \ln(\Omega(\eta)\Omega(\eta')) \right\}. \quad (49)$$

Comparison with (33) reveals,

$$\alpha \equiv \left(\frac{H}{2\pi}\right)^2 \left(\frac{H}{\sqrt{\pi}}\right)^{-\epsilon} \Gamma(2 - \frac{\epsilon}{2}), \quad \beta \equiv \frac{1}{4} \frac{\Gamma(3 - \epsilon)}{\Gamma(2 - \frac{\epsilon}{2})}, \quad (50)$$

and,

$$\gamma(y) \equiv \frac{1}{1 - \frac{\epsilon}{2}} \frac{1}{y^{1 - \frac{\epsilon}{2}}} + (1 - \frac{\epsilon}{4}) \left(\frac{1 - y^{\frac{\epsilon}{2}}}{\epsilon}\right) + \sum_{n=1}^{\infty} \left[\frac{1}{n} \frac{\Gamma(3 + n - \epsilon)}{\Gamma(2 + n - \frac{\epsilon}{2})} \frac{y^n}{4^{n+1}} - \frac{1}{n + \frac{\epsilon}{2}} \frac{\Gamma(3 + n - \frac{\epsilon}{2})}{(n+1)! \Gamma(2 - \frac{\epsilon}{2})} \frac{y^{n + \frac{\epsilon}{2}}}{4^{n+1}} \right]. \quad (51)$$

In this notation, the terms of the last bracket in Eq. (48) become,

$$- \frac{i}{2} \lambda i \Delta(x'; x') - i \delta m^2 = -i \lambda \alpha \beta \ln(\Omega(\eta')) . \quad (52)$$

Derivatives of the propagators can be written as,

$$\partial_\rho i \Delta(x; x') = \alpha \left\{ \gamma'(y) \frac{\partial y}{\partial x^\rho} + \beta \delta_\rho^0 H \Omega(\eta) \right\} . \quad (53)$$

These can be further reduced by noting,

$$\frac{\partial y}{\partial x^\rho} = H \Omega(\eta) \delta_\rho^0 y + 2 H^2 \Omega(\eta) \Omega(\eta') \Delta x_\rho . \quad (54)$$

At length one finds,

$$\begin{aligned} \partial_\rho i \Delta(x; x') \partial_\sigma i \Delta(x; x') &= \alpha^2 \left\{ 4 \Omega^2(\eta) \Omega^2(\eta') H^4 \Delta x_\rho \Delta x_\sigma \gamma'^2 \right. \\ &\quad + 2 \Omega^2(\eta) \Omega(\eta') H^3 \left[\delta_\rho^0 \Delta x_\sigma + \delta_\sigma^0 \Delta x_\rho \right] \left[y \gamma'^2 + \beta \gamma' \right] \\ &\quad \left. + \Omega^2(\eta) H^2 \delta_\rho^0 \delta_\sigma^0 \left[y^2 \gamma'^2 + 2 \beta \gamma' y + \beta^2 \right] \right\} . \end{aligned} \quad (55)$$

Expression (55) seems complicated due to the infinite sum in the definition of $\gamma(y)$. However, we need only retain terms which survive as $\epsilon \rightarrow 0$,

$$\lim_{\epsilon \rightarrow 0} \gamma'(y) = -\frac{1}{y^2} - \frac{1}{2y} , \quad (56)$$

$$\lim_{\epsilon \rightarrow 0} \left(\gamma'(y) \right)^2 = \frac{1}{y^{4-\epsilon}} + \frac{(2 - \frac{\epsilon}{2})}{2 y^{3-\epsilon}} + \frac{1}{4 y^2} . \quad (57)$$

One can therefore reduce (55) to,

$$\alpha^2 H^2 \Omega(\eta)^2 \left\{ H^2 \Omega(\eta')^2 \Delta x_\rho \Delta x_\sigma \left[\frac{4}{y^{4-\epsilon}} + \frac{2(2-\frac{\epsilon}{2})}{y^{3-\epsilon}} + \frac{1}{y^2} \right] \right. \\ \left. + H \Omega(\eta') \left[\delta_\rho^0 \Delta x_\sigma + \delta_\sigma^0 \Delta x_\rho \right] \left[\frac{2}{y^{3-\epsilon}} + \frac{1}{y^2} \right] + \delta_\rho^0 \delta_\sigma^0 \frac{1}{y^{2-\epsilon}} \right\}. \quad (58)$$

Note that we have retained the regularization for terms which can produce divergences in $T_{\mu\nu}^K(x)$.

It is useful to break $T_{\mu\nu}^K(x)$ into a sum of six terms of the general form,

$$I_{\mu\nu}^n(\eta) \equiv -i\lambda\alpha^3\beta H^2 \Omega^2(\eta) \left(\eta_{\mu\rho} \eta_{\nu\sigma} - \frac{1}{2} \eta_{\mu\nu} \eta_{\rho\sigma} \right) \\ \times \int d^D x' \Omega^{4-\epsilon}(\eta') \ln(\Omega(\eta')) F_n^{\rho\sigma}(x, x'). \quad (59)$$

The functions $F_n^{\rho\sigma}(x, x')$ are defined as follows:

$$F_1^{\rho\sigma}(x, x') = 4H^2 \Omega(\eta')^2 \Delta x^\rho \Delta x^\sigma \left[\frac{1}{y_{++}^{4-\epsilon}} - \frac{1}{y_{+-}^{4-\epsilon}} \right], \quad (60)$$

$$F_2^{\rho\sigma}(x, x') = 2(2 - \frac{\epsilon}{2}) H^2 \Omega(\eta')^2 \Delta x^\rho \Delta x^\sigma \left[\frac{1}{y_{++}^{3-\epsilon}} - \frac{1}{y_{+-}^{3-\epsilon}} \right], \quad (61)$$

$$F_3^{\rho\sigma}(x, x') = H^2 \Omega(\eta')^2 \Delta x^\rho \Delta x^\sigma \left[\frac{1}{y_{++}^2} - \frac{1}{y_{+-}^2} \right], \quad (62)$$

$$F_4^{\rho\sigma}(x, x') = 2H \Omega(\eta') [\delta_\rho^0 \Delta x^\sigma + \delta_\sigma^0 \Delta x^\rho] \left[\frac{1}{y_{++}^{3-\epsilon}} - \frac{1}{y_{+-}^{3-\epsilon}} \right], \quad (63)$$

$$F_5^{\rho\sigma}(x, x') = H \Omega(\eta') [\delta_\rho^0 \Delta x^\sigma + \delta_\sigma^0 \Delta x^\rho] \left[\frac{1}{y_{++}^2} - \frac{1}{y_{+-}^2} \right], \quad (64)$$

$$F_6^{\rho\sigma}(x, x') = \delta_\rho^0 \delta_\sigma^0 \left[\frac{1}{y_{++}^{2-\epsilon}} - \frac{1}{y_{+-}^{2-\epsilon}} \right]. \quad (65)$$

In the remainder of this section we will evaluate $I_{\mu\nu}^1(\eta)$ explicitly to illustrate the relevant techniques, and then simply give the final answers for the remaining five.

By removing the factors of Ω and H the first integral assumes the form,

$$I_{\mu\nu}^1(\eta) = -\frac{i\lambda H^{2-\epsilon}}{2^6 \pi^{6-\frac{3}{2}\epsilon}} \Omega^{-2+\epsilon}(\eta) \Gamma^2\left(2 - \frac{\epsilon}{2}\right) \Gamma(3 - \epsilon) \left[\delta_\mu^\rho \delta_\nu^\sigma - \frac{1}{2} \eta_{\mu\nu} \eta^{\rho\sigma} \right]$$

$$\times \int d^D x' \ln(\Omega(\eta')) \Omega^2(\eta') \left\{ \frac{\Delta x_\rho \Delta x_\sigma}{\Delta x_{++}^{8-2\epsilon}} - \frac{\Delta x_\rho \Delta x_\sigma}{\Delta x_{+-}^{8-2\epsilon}} \right\} . \quad (66)$$

Here, $\Delta x_{++}^2 \equiv \|\vec{x} - \vec{x}'\|^2 - (|\eta - \eta'| - ie)^2$, whereas $\Delta x_{+-}^2 \equiv \|\vec{x} - \vec{x}'\|^2 - (\eta - \eta' + ie)^2$. Now use the differential identities,

$$\partial_\rho \partial_\sigma \frac{1}{\Delta x^{4-2\epsilon}} = -(4-2\epsilon) \left[\frac{\eta_{\rho\sigma}}{\Delta x^{6-2\epsilon}} - (6-2\epsilon) \frac{\Delta x_\rho \Delta x_\sigma}{\Delta x^{8-2\epsilon}} \right] , \quad (67)$$

$$\partial^2 \frac{1}{\Delta x^{4-2\epsilon}} = 2(2-\epsilon)^2 \frac{1}{\Delta x^{6-2\epsilon}} , \quad (68)$$

$$\partial^2 \frac{1}{\Delta x^{2-2\epsilon}} = -2\epsilon(1-\epsilon) \frac{1}{\Delta x^{4-2\epsilon}} , \quad (69)$$

to write,

$$\frac{\Delta x_\rho \Delta x_\sigma}{\Delta x^{8-2\epsilon}} = -\frac{1}{8\epsilon(1-\epsilon)(2-\epsilon)(3-\epsilon)} \left[\partial_\rho \partial_\sigma + \frac{1}{2-\epsilon} \eta_{\rho\sigma} \partial^2 \right] \partial^2 \frac{1}{\Delta x^{2-2\epsilon}} . \quad (70)$$

These results pertain for both the $++$ and $+-$ terms. Since the range of x^μ integration does not depend upon x^μ , the derivatives can be pulled outside,

$$\begin{aligned} I_{\mu\nu}^1(\eta) &= \frac{i\lambda H^{2-\epsilon}}{2^9 \pi^{6-\frac{3}{2}\epsilon}} \Omega^{-2+\epsilon}(\eta) \frac{\Gamma^2(2-\frac{\epsilon}{2})\Gamma(2-\epsilon)}{(1-\epsilon)(3-\epsilon)} \left[\partial_\mu \partial_\nu - \eta_{\mu\nu} \partial^2 \right] \\ &\times \int d^D x' \ln(\Omega(\eta')) \Omega^2(\eta') \frac{\partial^2}{\epsilon} \left[\frac{1}{\Delta x_{++}^{2-2\epsilon}} - \frac{1}{\Delta x_{+-}^{2-2\epsilon}} \right] . \end{aligned} \quad (71)$$

It is at this stage that the parallel treatment of the $++$ and $+-$ ends. The crucial difference between the two terms is that Δx_{++} contains an absolute value, $|\eta - \eta'|$, whereas Δx_{+-} does not. Double derivatives of x_{++} can therefore result in a temporal delta function. Taking e to zero may or may not produce a factor of $\delta^D(x - x')$, depending upon how many powers of x_{++} there are. The unique power for general dimension $D = 4 - \epsilon$ turns out to be,

$$\partial^2 \frac{1}{\Delta x_{++}^{2-\epsilon}} = \frac{2\epsilon(2-\epsilon)i\delta(\Delta\eta)}{(\|\vec{x} - \vec{x}'\| + e^2)^{2-\frac{\epsilon}{2}}} \rightarrow 4i \frac{\pi^{2-\frac{\epsilon}{2}}}{\Gamma(1-\frac{\epsilon}{2})} \delta^D(x - x') , \quad (72)$$

In contrast the $+-$ term gives zero,

$$\partial^2 \frac{1}{\Delta x_{+-}^{2-\epsilon}} = 0 . \quad (73)$$

By using (73) we can write the $+-$ term in (71) in a form that remains finite in the unregulated limit,

$$\frac{\partial^2}{\epsilon} \left[\frac{1}{\Delta x_{+-}^{2-2\epsilon}} \right] = \frac{\partial^2}{\epsilon} \left[\frac{1}{\Delta x_{+-}^{2-2\epsilon}} - \frac{\mu^{-\epsilon}}{\Delta x_{+-}^{2-\epsilon}} \right]. \quad (74)$$

The arbitrary mass parameter μ has been introduced to maintain the correct dimensions. Since this term is manifestly ultraviolet finite we may as well take ϵ to zero,

$$\frac{\partial^2}{\epsilon} \left[\frac{1}{\Delta x_{+-}^{2-2\epsilon}} \right] \rightarrow \partial^2 \left[\frac{\ln(\mu^2 \Delta x_{+-}^2)}{2\Delta x_{+-}^2} \right]. \quad (75)$$

The analogous reduction of the $++$ term gives,

$$\frac{\partial^2}{\epsilon} \frac{1}{\Delta x_{++}^{2-2\epsilon}} = \frac{\partial^2}{\epsilon} \left[\frac{1}{\Delta x_{++}^{2-2\epsilon}} - \frac{\mu^{-\epsilon}}{\Delta x_{++}^{2-\epsilon}} \right] + \frac{4i\pi^{2-\frac{\epsilon}{2}}\mu^{-\epsilon}}{\epsilon\Gamma(1-\frac{\epsilon}{2})} \delta^D(x-x'), \quad (76)$$

$$\rightarrow \partial^2 \left[\frac{\ln(\mu^2 \Delta x_{++}^2)}{2\Delta x_{++}^2} \right] + \frac{4i\pi^{2-\frac{\epsilon}{2}}\mu^{-\epsilon}}{\epsilon\Gamma(1-\frac{\epsilon}{2})} \delta^D(x-x'). \quad (77)$$

Using (71), (75) and (77) we can bring the first integral to the form,

$$\begin{aligned} I_{\mu\nu}^1(\eta) &= \frac{i\lambda H^{2-\epsilon} \Omega^{-2+\epsilon}(\eta)}{2^9 \pi^{6-\frac{3}{2}\epsilon}} \frac{\Gamma(2-\epsilon)\Gamma(2-\frac{\epsilon}{2})}{(1-\epsilon)(3-\epsilon)} \left[\partial_\mu \partial_\nu - \eta_{\mu\nu} \partial^2 \right] \\ &\times \int d^d x' \ln(\Omega(\eta')) \Omega(\eta')^2 \left\{ \partial^2 \left[\left(\frac{\ln(\mu^2 \Delta x_{++}^2)}{2\Delta x_{++}^2} - \frac{\ln(\mu^2 \Delta x_{+-}^2)}{2\Delta x_{+-}^2} \right) \right] \right. \\ &\left. + \frac{4i\pi^{2-\frac{\epsilon}{2}}\mu^{-\epsilon}}{\epsilon\Gamma(1-\frac{\epsilon}{2})} \delta^D(x-x') \right\}. \end{aligned} \quad (78)$$

Now note that the integral has no dependence upon \vec{x} , so only the temporal derivatives matter. Evaluating δ -function integral and taking ϵ to zero in the finite terms gives,

$$\begin{aligned} I_{\mu\nu}^1(\eta) &= -\frac{\lambda H^{4-\epsilon} \Omega^{2+\epsilon}(\eta)}{2^7 \pi^{4-\epsilon}} \frac{(1-\frac{\epsilon}{2})\Gamma(1-\epsilon)\Gamma(2-\frac{\epsilon}{2})\mu^{-\epsilon}}{\epsilon(3-\epsilon)} \left[\delta_\mu^0 \delta_\nu^0 + \eta_{\mu\nu} \right] \\ &\times [6 \ln(\Omega(\eta)) + 5] + \frac{i\lambda H^2 \Omega^{-2}(\eta)}{2^{10} 3 \pi^6} \left[\delta_\mu^0 \delta_\nu^0 + \eta_{\mu\nu} \right] \\ &\times \partial_0^4 \int d^D x' \ln(\Omega(\eta')) \Omega^2(\eta') \left\{ \frac{\ln(\mu^2 \Delta x_{++}^2)}{\Delta x_{++}^2} - \frac{\ln(\mu^2 \Delta x_{+-}^2)}{\Delta x_{+-}^2} \right\}. \end{aligned} \quad (79)$$

To evaluate the integral in Eq. (79) we pull out another derivative using the identities:

$$\partial^2 \ln^2(\Delta x^2) = 8 \left[\frac{\ln(\Delta x^2)}{\Delta x^2} + \frac{1}{\Delta x^2} \right], \quad (80)$$

$$\partial^2 \ln(\Delta x^2) = \frac{4}{\Delta x^2}. \quad (81)$$

The remaining integrand possesses only logarithmic singularities. If we define the coordinate separations,

$$\Delta\eta \equiv \eta - \eta' \quad , \quad r \equiv \|\vec{x} - \vec{x}'\|, \quad (82)$$

the logarithms can be expanded as,

$$\ln[\mu^2 \Delta x_{++}^2] = \ln[\mu^2(\Delta\eta^2 - r^2)] + i\pi\theta(\Delta\eta^2 - r^2), \quad (83)$$

$$\ln[\mu^2 \Delta x_{+-}^2] = \ln[\mu^2(\Delta\eta^2 - r^2)] - i\pi\theta(\Delta\eta^2 - r^2). \quad (84)$$

Putting it all together gives the following reduction,

$$\begin{aligned} \partial_0^4 \int d^D x' \ln(\Omega(\eta')) \Omega^2(\eta') & \left\{ \frac{\ln(\mu^2 \Delta x_{++}^2)}{\Delta x_{++}^2} - \frac{\ln(\mu^2 \Delta x_{+-}^2)}{\Delta x_{+-}^2} \right\} \\ &= -i2\pi^2 \partial_0^6 \int_{-\frac{1}{H}}^{\eta} d\eta' \ln(\Omega(\eta')) \Omega^2(\eta') \int_0^{\Delta\eta} dr r^2 (\ln[\mu^2(\Delta\eta^2 - r^2)] - 1) \\ &= -i2\pi^2 \partial_0^6 \int_{-\frac{1}{H}}^{\eta} d\eta' \ln(\Omega(\eta')) \Omega^2(\eta') (\Delta\eta)^3 \left[\frac{2}{3} \ln(2\mu\Delta\eta) - \frac{11}{9} \right] \\ &= -i8\pi^2 \partial_0^3 \int_{-\frac{1}{H}}^{\eta} d\eta' \ln(\Omega(\eta')) \Omega^2(\eta') \ln(2\mu\Delta\eta). \end{aligned} \quad (85)$$

The lower limit of temporal integration at $\eta' = -H^{-1}$ (that is, $t' = 0$) derives from the fact that we release the state in free Bunch-Davies vacuum at this instant. I_1 can now be recast as:

$$\begin{aligned} I_{\mu\nu}^1(\eta) &= -\frac{\lambda H^4}{2^6 \pi^4} \Omega^2(\eta) [\delta_\mu^0 \delta_\nu^0 + \eta_{\mu\nu}] \left\{ \frac{\zeta}{\epsilon} \frac{\Omega^\epsilon(\eta)}{(3-\epsilon)} \left[3 \ln(\Omega(\eta)) + \frac{5}{2} \right] \right. \\ &\quad \left. + \frac{1}{6H^2 \Omega^4(\eta)} \partial_0^3 \int_{-\frac{1}{H}}^{\eta} d\eta' \ln(\Omega(\eta')) \Omega(\eta')^2 \ln(2\mu\Delta\eta) \right\}, \end{aligned} \quad (86)$$

where ζ is defined as

$$\zeta \equiv \left(\frac{\pi}{\mu H} \right)^\epsilon \left(1 - \frac{\epsilon}{2} \right)^2 \Gamma(1 - \epsilon) \Gamma \left(1 - \frac{\epsilon}{2} \right) . \quad (87)$$

To evaluate the integral in Eq. (86), we change variables to $\Omega(\eta') = -\frac{1}{H\eta'}$ and expand the logarithm,

$$\begin{aligned} & \frac{1}{6H^2\Omega^4(\eta)} \partial_0^3 \int_{-\frac{1}{H}}^\eta d\eta' \ln(\Omega(\eta')) \Omega^2(\eta') \ln(2\mu\Delta\eta) \\ &= \frac{1}{6\Omega^4} \left(\Omega^2 \frac{\partial}{\partial \Omega} \right)^2 \left[\Omega^2 \ln(\Omega) \left[\ln\left(\frac{2\mu}{H}\right) - \ln(\Omega) \right] \right] \\ & \quad - \frac{1}{6\Omega^4} \left(\Omega^2 \frac{\partial}{\partial \Omega} \right)^3 \sum_n \frac{1}{n\Omega^n} \int_1^\Omega d\Omega' \Omega'^n \ln(\Omega') \end{aligned} \quad (88)$$

$$\begin{aligned} &= -\ln^2(\Omega) - \frac{8}{3} \ln(\Omega) + \ln\left(\frac{2\mu}{H}\right) \ln(\Omega) + \frac{5}{6} \ln\left(\frac{2\mu}{H}\right) \\ & \quad - \frac{1}{6} \left[1 + \pi^2 - \sum_n \frac{(n-1)(n-2)}{(n+1)^2} \Omega^{-n-1} \right] . \end{aligned} \quad (89)$$

Our final result for $I_{\mu\nu}^1(\eta)$ is,

$$\begin{aligned} I_{\mu\nu}^1(\eta) &= -\frac{\lambda H^4}{2^6 \pi^4} \Omega^2 \left[\delta_\mu^0 \delta_\nu^0 + \eta_{\mu\nu} \right] \left\{ \left[\frac{\zeta}{\epsilon} + \ln\left(\frac{2\mu}{H}\right) - \frac{3}{2} \right] \ln(\Omega) \right. \\ & \quad \left. + \frac{5}{6} \left[\frac{\zeta}{\epsilon} + \ln\left(\frac{2\mu}{H}\right) \right] + \frac{1}{9} - \frac{\pi^2}{6} + \frac{1}{6} \sum_{n=1}^\infty \frac{(n-1)(n-2)}{(n+1)^2} \Omega^{-n-1} \right\} . \end{aligned} \quad (90)$$

The five other terms in Eq. (59) can be evaluated similarly. The results are,

$$\begin{aligned} I_{\mu\nu}^2(\eta) &= -\frac{i\lambda H^{4-\epsilon}}{2^7 \pi^{6-\frac{3}{2}\epsilon}} \Omega^{-1+\epsilon} \Gamma(2 - \frac{\epsilon}{2}) \Gamma(3 - \epsilon) \Gamma(3 - \frac{\epsilon}{2}) \left[\delta_\mu^\rho \delta_\nu^\sigma - \frac{1}{2} \eta_{\mu\nu} \eta^{\rho\sigma} \right] \\ & \quad \times \int d^D x' \ln(\Omega(\eta')) \Omega^3(\eta') \left\{ \frac{\Delta x_\rho \Delta x_\sigma}{\Delta x_{++}^{6-2\epsilon}} - \frac{\Delta x_\rho \Delta x_\sigma}{\Delta x_{+-}^{6-2\epsilon}} \right\} \\ &= \frac{\lambda H^4}{2^6 \pi^4} \Omega^2 \left\{ \eta_{\mu\nu} \left[\left[\frac{\zeta}{\epsilon} + \ln\left(\frac{2\mu}{H}\right) - \frac{9}{4} \right] \ln(\Omega) + \frac{5}{4} - \frac{\pi^2}{6} \right. \right. \\ & \quad \left. \left. + \sum_{n=2}^\infty \frac{\Omega^{-n-1}}{(n+1)^2} \right] - \left[\delta_\mu^0 \delta_\nu^0 + \frac{1}{2} \eta_{\mu\nu} \right] \ln(\Omega) \right\} . \end{aligned} \quad (91)$$

$$\begin{aligned}
I_{\mu\nu}^3(\eta) &= -\frac{i\lambda H^{6-3\epsilon}}{2^8\pi^{6-\frac{3}{2}\epsilon}}\Gamma^2(2-\frac{\epsilon}{2})\Gamma(3-\epsilon)\left[\delta_\mu{}^\rho\delta_\nu{}^\sigma-\frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}\right] \\
&\quad \times \int d^D x' \ln(\Omega(\eta'))\Omega^{4-\epsilon}(\eta')\left\{\frac{\Delta x_\rho\Delta x_\sigma}{\Delta x_{++}^4}-\frac{\Delta x_\rho\Delta x_\sigma}{\Delta x_{+-}^4}\right\} \\
&= -\frac{\lambda H^4}{2^6\pi^4}\Omega^2\delta_\mu{}^0\delta_\nu{}^0\left\{\frac{1}{3}\ln(\Omega)-\frac{5}{18}+\frac{1}{2}\Omega^{-2}-\frac{2}{9}\Omega^{-3}\right\}. \quad (92)
\end{aligned}$$

$$\begin{aligned}
I_{\mu\nu}^4(\eta) &= -\frac{i\lambda H^{3-\epsilon}}{2^7\pi^{6-\frac{3}{2}\epsilon}}\Omega^{-1+\epsilon}\Gamma^2(2-\frac{\epsilon}{2})\Gamma(3-\epsilon)\left[\delta_\mu{}^\rho\delta_\nu{}^\sigma-\frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}\right] \\
&\quad \times \int d^D x' \ln(\Omega(\eta'))\Omega^2(\eta')\left[\delta_\rho{}^0\Delta x_\sigma+\delta_\sigma{}^0\Delta x_\rho\right]\left\{\frac{1}{\Delta x_{++}^{6-2\epsilon}}-\frac{1}{\Delta x_{+-}^{6-2\epsilon}}\right\} \\
&= \frac{\lambda H^4}{2^6\pi^4}\Omega^2(\eta)\left[\delta_\mu{}^0\delta_\nu{}^0+\frac{1}{2}\eta_{\mu\nu}\right]\left\{\left[\frac{\zeta}{\epsilon}+\ln\left(\frac{2\mu}{H}\right)-\frac{3}{2}\right]2\ln(\Omega)\right. \\
&\quad \left.+\frac{\zeta}{\epsilon}+\ln\left(\frac{2\mu}{H}\right)+1-\frac{\pi^2}{3}-\sum_{n=1}^{\infty}\frac{(n-1)}{(n+1)^2}\Omega^{-n-1}\right\}. \quad (93)
\end{aligned}$$

$$\begin{aligned}
I_{\mu\nu}^5(\eta) &= -\frac{i\lambda H^{5-3\epsilon}}{2^8\pi^{6-\frac{3}{2}\epsilon}}\Gamma^2(2-\frac{\epsilon}{2})\Gamma(3-\epsilon)\left[\delta_\mu{}^\rho\delta_\nu{}^\sigma-\frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}\right] \\
&\quad \times \int d^D x' \ln(\Omega(\eta'))\Omega^{3-\epsilon}(\eta')\left[\delta_\rho{}^0\Delta x_\sigma+\delta_\sigma{}^0\Delta x_\rho\right]\left\{\frac{1}{\Delta x_{++}^4}-\frac{1}{\Delta x_{+-}^4}\right\} \\
&= \frac{\lambda H^4}{2^6\pi^4}\Omega^2\left[\delta_\mu{}^0\delta_\nu{}^0+\frac{1}{2}\eta_{\mu\nu}\right]\left\{\ln(\Omega)-\frac{1}{2}+\frac{1}{2}\Omega^{-2}\right\}. \quad (94)
\end{aligned}$$

$$\begin{aligned}
I_{\mu\nu}^6(\eta) &= -\frac{i\lambda H^{4-\epsilon}}{2^8\pi^{6-\frac{3}{2}\epsilon}}\Omega^\epsilon\Gamma^2(2-\frac{\epsilon}{2})\Gamma(3-\epsilon)\left[\delta_\mu{}^0\delta_\nu{}^0+\frac{1}{2}\eta_{\mu\nu}\right] \\
&\quad \times \int d^D x' \ln(\Omega(\eta'))\Omega^2(\eta')\left\{\frac{1}{\Delta x_{++}^{4-2\epsilon}}-\frac{1}{\Delta x_{+-}^{4-2\epsilon}}\right\} \\
&= -\frac{\lambda H^4}{2^6\pi^4}\Omega^2\left[\delta_\mu{}^0\delta_\nu{}^0+\frac{1}{2}\eta_{\mu\nu}\right]\left\{\left[\frac{\zeta}{\epsilon}+\ln\left(\frac{2\mu}{H}\right)-\frac{3}{2}\right]\ln(\Omega)\right. \\
&\quad \left.+1-\frac{\pi^2}{6}+\sum_{n=1}^{\infty}\frac{\Omega^{-n-1}}{(n+1)^2}\right\}. \quad (95)
\end{aligned}$$

Summing the six terms gives the total order λ contribution from the kinetic energy,

$$T_{\mu\nu}^K(x) = \frac{\lambda H^4}{2^6\pi^4}\Omega^2\left\{\delta_\mu{}^0\delta_\nu{}^0\left[\frac{1}{6}\left(\frac{\zeta}{\epsilon}+\ln\left(\frac{2\mu}{H}\right)\right)-\frac{1}{3}\ln(\Omega)\right]\right.$$

$$\begin{aligned}
& -\frac{1}{3} + \frac{2}{9}\Omega^{-3} - \frac{1}{6} \sum_{n=1}^{\infty} \frac{(n+2)}{(n+1)} \Omega^{-n-1} \Big] + \eta_{\mu\nu} \left[\left(\frac{\zeta}{\epsilon} + \ln \left(\frac{2\mu}{H} \right) \right) \times \right. \\
& \left. \left(\frac{1}{2} \ln(\Omega) - \frac{1}{3} \right) - \frac{3}{2} \ln(\Omega) + \frac{8}{9} - \frac{\pi^2}{12} - \frac{1}{6} \sum_{n=1}^{\infty} \frac{(n^2-4)}{(n+1)^2} \Omega^{-n-1} \right] \Big\}. \quad (96)
\end{aligned}$$

We therefore obtain the following order λ kinetic energy density,

$$\begin{aligned}
\rho_K = \frac{\lambda H^4}{2^6 \pi^4} \Big\{ \left[\frac{\zeta}{\epsilon} + \ln \left(\frac{2\mu}{H} \right) \right] \left[-\frac{1}{2} \ln(\Omega) + \frac{1}{2} \right] + \frac{7}{6} \ln(\Omega) - \frac{11}{9} \\
+ \frac{\pi^2}{12} + \frac{2}{9} \Omega^{-3} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{(n+2)}{(n+1)^2} \Omega^{-n-1} \Big\}, \quad (97)
\end{aligned}$$

and pressure,

$$\begin{aligned}
p_K = \frac{\lambda H^4}{2^6 \pi^4} \Big\{ \left[\frac{\zeta}{\epsilon} + \ln \left(\frac{2\mu}{H} \right) \right] \left[\frac{1}{2} \ln(\Omega) - \frac{1}{3} \right] - \frac{3}{2} \ln(\Omega) + \frac{8}{9} - \frac{\pi^2}{12} \\
- \frac{1}{6} \sum_{n=1}^{\infty} \frac{(n^2-4)}{(n+1)^2} \Omega^{-n-1} \Big\}. \quad (98)
\end{aligned}$$

7 The conformal counterterm

Since $\zeta = 1 + O(\epsilon)$, $T_{\mu\nu}^K$ contains a divergence proportional to $g_{\mu\nu} \ln(\Omega)/\epsilon$. This is an overlapping divergence, and it must be canceled if the stress-energy tensor is to be a well defined operator. It cannot be absorbed into a renormalization of the cosmological constant on account of the factor of $\ln(\Omega)$. The difficulty is resolved by the conformal counterterm (15),³

$$\begin{aligned}
T_{\mu\nu}^C(x) = -2\delta\xi \Big\{ (D-1)H^2 g_{\mu\nu}(x) i\Delta(x; x) \\
+ \left[g_{\mu\nu}(x) g^{\rho\sigma}(x) - \delta^\rho_\mu \delta^\sigma_\nu \right] \left(\partial_\rho \partial_\sigma - \Gamma^\alpha_{\rho\sigma}(x) \partial_\alpha \right) i\Delta(x; x) \Big\}. \quad (99)
\end{aligned}$$

(See the first of the graphs on Fig. 4.) Recall that this comes from a term in the action which is contrived to vanish in de Sitter background so it affects only the stress tensor and makes no contribution to purely scalar processes.

The affine connection for de Sitter conformal coordinates is,

$$\Gamma^\rho_{\mu\nu} = H\Omega(\eta) \left[\delta^\rho_\mu \delta^0_\nu + \delta^\rho_\nu \delta^0_\mu - \eta_{\mu\nu} \eta^{\rho 0} \right]. \quad (100)$$

³The potential problem and its resolution were adumbrated in previous work [8].

Our simple, D -dimensional result for the coincidence limit (34) of the propagator implies the following result for the covariant derivative,

$$\left(\partial_\rho \partial_\sigma - \Gamma^\alpha_{\rho\sigma}(x) \partial_\alpha\right) i\Delta(x; x) = -\frac{H^{4-\epsilon}}{2^3 \pi^{2-\epsilon/2}} \Gamma(3-\epsilon) \left[g_{\mu\nu} + \delta_\mu^0 \delta_\nu^0 \Omega^2(\eta)\right]. \quad (101)$$

Substitution and a few trivial manipulations yield,

$$T_{\mu\nu}^C(x) = -\frac{\delta\xi H^{4-\epsilon}}{2^2 \pi^{2-\epsilon/2}} \left\{ \left[\left(\frac{3-\epsilon}{\epsilon}\right) \Gamma\left(3-\frac{\epsilon}{2}\right) + \Gamma(4-\epsilon) \left[\ln(\Omega) - 1\right] \right] g_{\mu\nu} + \Gamma(3-\epsilon) \left[g_{\mu\nu} + \delta_\mu^0 \delta_\nu^0 \Omega^2\right] \right\}. \quad (102)$$

It is straightforward to extract the energy density and pressure,

$$\rho_C = \frac{\delta\xi H^{4-\epsilon}}{2^2 \pi^{2-\epsilon/2}} \left\{ \left(\frac{3-\epsilon}{\epsilon}\right) \Gamma\left(3-\frac{\epsilon}{2}\right) + \Gamma(4-\epsilon) (\ln(\Omega) - 1) \right\}, \quad (103)$$

$$p_C = -\frac{\delta\xi H^{4-\epsilon}}{2^2 \pi^{2-\epsilon/2}} \left\{ \left(\frac{3-\epsilon}{\epsilon}\right) \Gamma\left(3-\frac{\epsilon}{2}\right) + \Gamma(4-\epsilon) (\ln(\Omega) - 1) + \Gamma(3-\epsilon) \right\}. \quad (104)$$

8 The fully renormalized result

Summing the potential and kinetic energy densities (47,97) gives,

$$\begin{aligned} \rho_{PK} = & \frac{\lambda H^4}{2^6 \pi^4} \left\{ -\left(\frac{\pi}{H^2}\right)^\epsilon \frac{\Gamma^2(3-\frac{\epsilon}{2})}{8\epsilon^2} + \left[\frac{\zeta}{\epsilon} + \ln\left(\frac{2\mu}{H}\right)\right] \left[\frac{1}{2} - \frac{1}{2} \ln(\Omega(\eta))\right] \right. \\ & + \frac{1}{2} \ln^2(\Omega(\eta)) + \frac{7}{6} \ln(\Omega(\eta)) - \frac{11}{9} + \frac{\pi^2}{12} + \frac{2}{9} \Omega(\eta)^{-3} \\ & \left. - \frac{1}{2} \sum_{n=1}^{\infty} \frac{(n+2)}{(n+1)^2} \Omega(\eta)^{-n-1} \right\}, \end{aligned} \quad (105)$$

Doing the same for the potential and kinetic pressures (47,98) yields,

$$p_{PK} = \frac{\lambda H^4}{2^6 \pi^4} \left\{ \left(\frac{\pi}{H^2}\right)^\epsilon \frac{\Gamma^2(3-\frac{\epsilon}{2})}{8\epsilon^2} + \left[\frac{\zeta}{\epsilon} + \ln\left(\frac{2\mu}{H}\right)\right] \left[\frac{1}{2} \ln(\Omega(\eta)) - \frac{1}{3}\right] \right\}$$

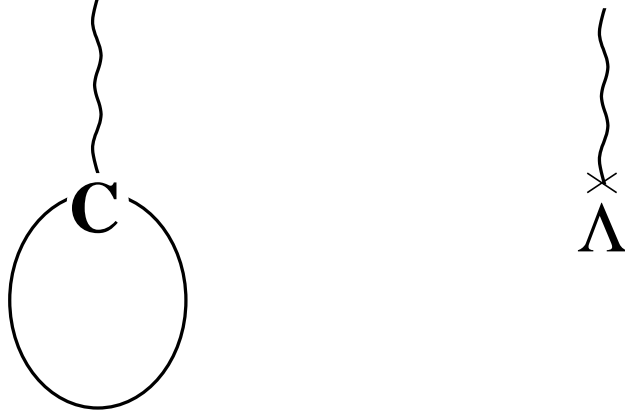


Figure 4: Contributions to the scalar stress-energy tensor at order λ from the conformal counterterm C and the counterterm for the bare cosmological constant Λ .

$$\begin{aligned}
& -\frac{1}{2} \ln^2(\Omega(\eta)) - \frac{3}{2} \ln(\Omega(\eta)) + \frac{8}{9} - \frac{\pi^2}{12} \\
& -\frac{1}{6} \sum_{n=1}^{\infty} \frac{(n^2 - 4)}{(n+1)^2} \Omega(\eta)^{-n-1} \Big\} .
\end{aligned} \tag{106}$$

It is an important check of the calculation that (105) and (106) do satisfy the D-dimensional conservation law:

$$\dot{\rho}_{PK} = -(3 - \epsilon)H(\rho_{PK} + p_{PK}) . \tag{107}$$

Although conserved, the potential and kinetic terms still harbor divergences. We eliminate the nonlocal, “overlapping” divergence by choosing the divergent part of the conformal counterterm thus,

$$\delta\xi \equiv \frac{\lambda}{2^4\pi^2} \left(\frac{H^2}{\pi} \right)^{\epsilon/2} \left\{ \frac{\zeta}{2\epsilon\Gamma(4-\epsilon)} + \delta\xi_{\text{fnt}} \right\} . \tag{108}$$

We will choose the finite part $\delta\xi_{\text{fnt}}$ later so as to achieve maximum simplicity in the fully renormalized result. The remaining divergences are proportional to $g_{\mu\nu}$ so they can be absorbed by a cosmological counterterm. The natural choice is,

$$\frac{\delta\Lambda}{8\pi G} = \frac{\lambda H^4}{2^6\pi^4} \left\{ \left(\frac{\pi}{H^2} \right)^{\epsilon} \frac{\Gamma^2(3 - \frac{\epsilon}{2})}{8\epsilon^2} - \frac{\zeta}{2\epsilon^2} \frac{\Gamma(3 - \frac{\epsilon}{2})}{\Gamma(3 - \epsilon)} \right\}$$

$$- \left(\frac{3-\epsilon}{\epsilon} \right) \Gamma \left(3 - \frac{\epsilon}{2} \right) \delta \xi_{\text{fnt}} + \delta \Lambda_{\text{fnt}} \Big\} , \quad (109)$$

where we leave the finite part for later determination.

Because the conformal and cosmological counterterms cancel all ultraviolet divergences we can take ϵ to zero in the renormalized energy density and pressure,

$$\begin{aligned} \rho_{\text{ren}} = & \frac{\lambda H^4}{2^6 \pi^4} \left\{ \frac{1}{2} \ln^2(\Omega) + \left[\frac{7}{6} - \frac{1}{2} \ln \left(\frac{2\mu}{H} \right) + 6\delta \xi_{\text{fnt}} \right] \ln(\Omega) - \frac{11}{9} + \frac{\pi^2}{12} \right. \\ & \left. + \frac{1}{2} \ln \left(\frac{2\mu}{H} \right) - 6\delta \xi_{\text{fnt}} + \delta \Lambda_{\text{fnt}} + \frac{2}{9} \Omega(\eta)^{-3} - \frac{1}{2} \sum_n \frac{n+2}{(n+1)^2} \Omega^{-n-1} \right\} , \quad (110) \end{aligned}$$

$$\begin{aligned} p_{\text{ren}} = & -\frac{\lambda H^4}{2^6 \pi^4} \left\{ \frac{1}{2} \ln^2(\Omega) + \left[\frac{3}{2} - \frac{1}{2} \ln \left(\frac{2\mu}{H} \right) + 6\delta \xi_{\text{fnt}} \right] \ln(\Omega) - \frac{5}{6} \right. \\ & \left. + \frac{\pi^2}{12} + \frac{1}{3} \ln \left(\frac{2\mu}{H} \right) - 4\delta \xi_{\text{fnt}} + \delta \Lambda_{\text{fnt}} + \frac{1}{6} \sum_n \frac{(n^2-4)}{(n+1)^2} \Omega^{-n-1} \right\} . \quad (111) \end{aligned}$$

The result can be considerably simplified by making a suitable choice for the finite parts of the counterterms,

$$\delta \xi_{\text{fnt}} = -\frac{7}{36} + \frac{1}{12} \ln \left(\frac{2\mu}{H} \right) \quad , \quad \delta \Lambda_{\text{fnt}} = \frac{1}{18} - \frac{\pi^2}{12} . \quad (112)$$

The final results are,

$$\rho_{\text{ren}} = \frac{\lambda H^4}{2^6 \pi^4} \left\{ \frac{1}{2} \ln^2(\Omega) + \frac{2}{9} \Omega^{-3} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{n+2}{(n+1)^2} \Omega^{-n-1} \right\} , \quad (113)$$

$$p_{\text{ren}} = -\frac{\lambda H^4}{2^6 \pi^4} \left\{ \frac{1}{2} \ln^2(\Omega) + \frac{1}{3} \ln(\Omega) + \frac{1}{6} \sum_{n=1}^{\infty} \frac{n^2-4}{(n+1)^2} \Omega^{-n-1} \right\} . \quad (114)$$

We have verified the conjecture of ref. [8] that the kinetic contributions are subdominant to the potential contributions by one factor of $\ln(\Omega)$. This suffices to prove that the model violates the Weak Energy Condition on cosmological scales,

$$\rho_{\text{ren}} + p_{\text{ren}} = \frac{\lambda H^4}{2^6 \pi^4} \left\{ -\frac{1}{3} \ln(\Omega) + \frac{2}{9} \Omega^{-3} - \frac{1}{6} \sum_{n=1}^{\infty} \frac{n+2}{n+1} \Omega^{-n-1} \right\} . \quad (115)$$

The leading term of this expression is exactly that predicted in equation (49) of ref. [8]. The difference is that we have rigorously proved it. We also have the sub-dominant corrections *and* we have a procedure that can be pushed to arbitrarily high order.

Two comments are in order concerning (115). First, there is a logarithmic singularity at $\ln(\Omega) = Ht = 0$ whereas the correct result is that the stress-energy tensor vanishes then. The problem seems to be caused by assuming $\ln(\Omega) \gg \epsilon$ at various places in the reduction. This means we should not trust the result for times infinitesimally close to $Ht = 0$ but it should be completely reliable at later times.

The second comment has to do with the length of time during which we can expect the Weak Energy Condition to be violated. The ϕ^4 potential obviously has some tendency to force the field back down. One might expect this classical effect to eventually balance against the quantum uncertainty pressure to result in a constant energy density and pressure.⁴ We can estimate how long the system takes to reach this stage by asking when higher order effects become comparable with the two loop result.

Consider a diagram with $2N$ external scalar lines. At L loop order the number of ϕ^4 interaction vertices is,

$$V = L + N - 1 . \quad (116)$$

Each contributes a factor of λ , so the diagram goes like λ^{L+N-1} . The number of internal propagators is,

$$P = 2L + N - 2 . \quad (117)$$

Since we are computing a Schwinger diagram, there will be V cancellations between $+$ and $-$ variations, which give the θ -function imaginary part of the logarithm. However, there are also V temporal integrations, each one of which can potentially result in an extra factor of $\ln(\Omega)$. Hence the strongest possible effect for the $2N$ -point vertex at L loop order is,

$$V_{2N}^L \sim \lambda^{L+N-1} \left(\ln(\Omega) \right)^{2L+N-2} . \quad (118)$$

⁴However, one should be alive to the possibility that stochastic effects result in the scalar's further migration up its potential in certain portions of the universe [21].

The stress-energy tensor corresponds to $N = 0$ scalar lines so the dominant contribution at L loop order is,

$$T_{\mu\nu}^{L-loop} \sim g_{\mu\nu} H^4 \left(\lambda \ln^2(\Omega) \right)^{L-1} . \quad (119)$$

It follows that perturbation theory breaks down when $\ln(\Omega) \sim 1/\sqrt{\lambda}$. Since λ is assumed small we see that the phase of super-acceleration can be made to last for an enormous number of e-foldings. Note that all the higher point diagrams remain perturbatively weak during this entire period,

$$\lim_{Ht \rightarrow \lambda^{-1/2}} V_{2N}^L \sim \lambda^{N/2} . \quad (120)$$

It should therefore be valid to use perturbation theory almost up to $\ln(\Omega) = 1/\sqrt{\lambda}$.

Note Added: After the completion of this work we became aware of a highly significant paper by Starobinsky and Yokoyama [22] in which stochastic techniques are employed to sum the leading powers of Ht at each order.

Acknowledgments

It is a pleasure to acknowledge stimulating and informative conversations with L. R. Abramo. It should also be noted that we have used some of the figure files he prepared for ref. [8]. This work was partially supported by DOE contract DE-FG02-97ER41029 and by the Institute for Fundamental Theory.

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